$$xv+1=\frac{\{(2n-1)(2n+1)[2n(2n-1)-1][2n(2n+1)-1]-2n(8n^2-1)\}^2}{(s-1)^2};$$

$$yv+1=\frac{\{(2n-1)(2n+1)[2n(2n-1)-1][2n(2n+1)-1]+2n(8n^2-1)\}^2}{(s-1)^2};$$

$$zv+1=\frac{\{16n^4(2n-1)(2n+1)-(8n^2-1)\}^2}{(s-1)^2};\text{ and}$$

$$wv+1=\frac{\{24n^2(2n^2-1)(2n-1)(2n+1)-1\}^2}{(s-1)^2}.$$

**74.** Proposed by O. W. ANTHONY, M. Sc., Instructor in Mathematics, Boys' High School, New York City.

Solve $x^2+y^2=\square$, $z^2+w^2=\square$, $y^2+w^2=\square$.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Take any two integral equations in which the sum of two squares equals a square, as

$$a^2+b^2=c^2,\text{ and } a_1^2+b_1^2=c_1^2.$$

Multiply the terms of the first equation by the first term and the second term, respectively, of the second equation. Also multiply the terms of the second equation by the first term and the second term, respectively, of the first equation. We then have

$$(aa_1)^2+(a_1b)^2=(a_1c)^2\ \ldots.(1),$$

$$(ab_1)^2+(bb_1)^2=(b_1c)^2\ \ldots.(2),$$

$$(aa_1)^2+(ab_1)^2=(ac_1)^2\ \ldots.(3),$$

$$(a_1b)^2+(bb_1)^2=(bc_1)^2\ \ldots\ (4).$$

Now put $x=aa_1$, $y=a_1b$, $z=ab_1$, and $w=bb_1$; then equations (1), (2), and (4) are the three required by the problem, there being added, in the solution, $x^2+z^2=\square\ \ldots.(3)$.

By means of the formula $(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2$, find a few integral numerical equations.

Take $m=2$, $n=1$; then $4^2+3^2=5^2\ \ldots.(1)$.

Take $m=3$, $n=2$; then $12^2+5^2=13^2\ \ldots.(2)$.

Take $m=4$, $n=1$; then $8^2+15^2=17^2\ \ldots.(3)$.

Take $m=5$, $n=2$; then $20^2+21^2=29^2\cdots\cdot(4)$, etc.

From (1) and (2), $x=48$, $y=36$, $z=20$, $w=15$.

From (1) and (3), $x=32$, $y=24$, $z=60$, $w=45$.

From (1) and (4), $x=80$, $y=60$, $z=84$, $w=63$.

From (2) and (3), $x=96$, $y=40$, $z=180$, $w=75$, etc.

**II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

Take $$(2mn)^2+(m^2-n^2)^2=(m^2+n^2)^2,$$

and $$(2pq)^2+(p^2-q^2)^2=(p^2+q^2)^2,$$

two general equations for finding the sum of two squares equal to a square.

Multiply the terms of the first equation by the first term and second term, respectively, of the second equation. Also multiply the terms of the second equation by the first term and the second term, respectively, of the first equation.

Then, $[4mnpq]^2+[2pq(m^2-n^2)]^2=[2pq(m^2+n^2)]^2\ldots.(1)$.

$[2mn(p^2-q^2)]^2+[(m^2-n^2)(p^2-q^2)]^2=[(m^2+n^2)(p^2-q^2)]^2\ldots.(2)$,

$[4mnpq]^2+[2mn(p^2-q^2)]^2=[2mn(p^2+q^2)]^2\ldots.(3)$,

$[2pq(m^2-n^2)]^2+[(m^2-n^2)(p^2-q^2)]^2=[(m^2-n^2)(p^2+q^2)]^2\ldots.(4)$.

Now put $x=4mnpq$, $y=2pq(m^2-n^2)$, $z=2mn(p^2-q^2)$, and $w=(m^2-n^2)(p^2-q^2)$, and the general equations (1), (2), and (4) are the three required by the problem ; $x^2+z^2=\square$, as equation (3), being added in the solution.

$m$, $n$, $p$, and $q$ are any integers, $m>n$, and $p>q$, and to obtain different values for $x$, $y$, $z$, and $w$, not more than two values, assigned to $m$, $n$, $p$, and $q$ may be alike.

Take $m=2$, $n=1$, $p=3$, $q=2$ ; then $x=48$, $y=36$, $z=20$, $w=15$.

Take $m=2$, $n=1$, $p=3$, $q=1$ ; then $x=24$, $y=18$, $z=32$, $w=24$.

Take $m=4$, $n=3$, $p=3$, $q=2$ ; then $x=288$, $y=84$, $z=120$, $w=35$, etc.

**III. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

$y$ and $w$ only, of the four quantities, are used twice. Assign to $y$ a numerical value such that it is found at least in *two sets* of the sum of two squares equal to a square.

(A)......Put $y=8$. We find $8^2+6^2=10^2$, and $8^2+15^2=17^2$.

(1). Take $x=6$ and $w=15$.

We find $15^2+8^2=17^2$, $15^2+20^2=25^2$, $15^2+36^2=39^2$, and $15^2+112^2=113^2$. Then $z=20$, 36, or 112.

(B)......Put $y=12$. We find $12^2+5^2=13^2$, $12^2+9^2=15^2$, $12^2+16^2=20^2$, and $12^2+35^2=37^2$.

(1). Take $x=5$, and $w=9$.

We find $9^2+12^2=15^2$, and $9^2+40^2=41^2$. Then $z=40$.

(2). Take $x=5$, and $w=16$.

We find $16^2+12^2=20^2$, $16^2+30^2=34^2$, and $16^2+63^2=65^2$. Then $z=30$ or 63.

(3). Take $x=5$, and $w=35$.

We find $35^2+12^2=37^2$, $35^2+84^2=91^2$, $35^2+120^2=125^2$, and $35^2+612^2=613^2$. Then $z=84$, 120, or 612.

(4). Take $x=9$, and $w=16$.

Then from (2), $z=30$, or 63. And so on.

IV. **Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.**

In problems where $x^2+y^2=\square$, $z^2+w^2=\square$, $x^2+z^2=\square$, and $y^2+w^2=\square$, we have the proportion $x:y=z:w$.

Now take two integers the sum of whose squares equals a square, and arrange them in an identical proportion.

Then take two integers of the same kind and arrange them, underneath the first proportion, in an identical proportion of alternation as compared with the first proportion.

Then find the products, term by term, of these two proportions ; and the four products will be the required numbers.

Take $3^2+4^2=5^2$, and $5^2+12^2=12^2$.

$$\begin{array}{c} x:y=z:w \\ \hline 3:4=3:4 \\ 3:5=12:12 \\ \hline 15:20=36:48 \end{array}$$

$15^2+20^2=25^2$, $36^2+48^2=60^2$, $15^2+36^2=39^2$, $20^2+48^2=52^2$.

V. **Solution by J. H. DRUMMOND, LL. D., Portland, Me.**

Manifestly $x$ and $y$, and $z$ and $w$, are the bases and perpendiculars of two different right-angled triangles. Hence $x=m^2-n^2$, and $y=2mn$; and $z=p(m^2-n^2)$, and $w=2pmn$. But $y^2+w^2=\square$. Or $4p^2m^2n^2+4m^2n^2=\square$, or $p^2+1=\square=$(say) $(pq-1)^2$. From which $p=\dfrac{2q}{q^2-1}$ Then $z=\dfrac{2q(m^2-n^2)}{q^2-1}$, and $w=\dfrac{2qmn}{q^2-1}$, in which $m$, $n$, and $q$ may be any numbers, $q>1$, and $m>n$.

Also solved by *A. H. BELL, CHARLES C. CROSS, ELMER SCHUYLER*, and *G. B. M. ZERR.*

## PROBLEMS FOR SOLUTION.

### ARITHMETIC.

112. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

The cost of an article is $\$4.\dfrac{.\dot{2}9\dot{7}}{1.00\dot{3}}$. The selling price is $\$6.\frac{1000}{33337}$. What is the gain % ?

113. **Proposed by B. F. SINE, Principal of Normal School, Capon Bridge, W. Va.**

In what time can a note of \$5280, bearing 6% interest, be paid by paying \$600 a year ? [Solve by arithmetic].

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.